Periodic Billiard Paths in Triangles

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Abstract

Let a point move on a frictionless plane bounded by a closed figure. If it hits the boundary, it changes its direction of motion such that the angle of reflection is equal to the angle of incidence. The path that the point follows is called a billiard path. If the point returns to a location with the same direction of motion it had before at that location, then its path is said to be periodic. While it is known that a periodic billiard path exists in every acute triangle and in every right triangle, it is an open problem whether or not every obtuse triangle has a periodic billiard path.

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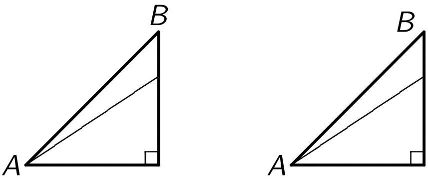
- A *perpendicular* path is one whose point hits an edge at a right angle

Unfolding



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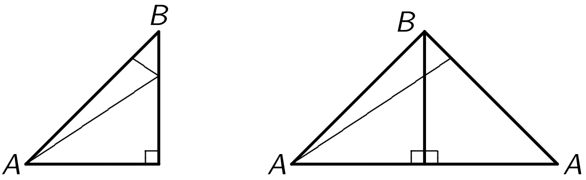

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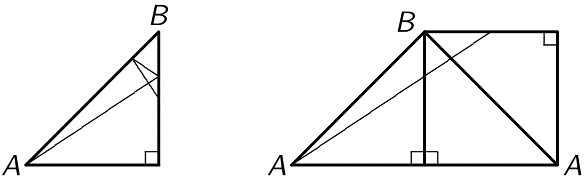

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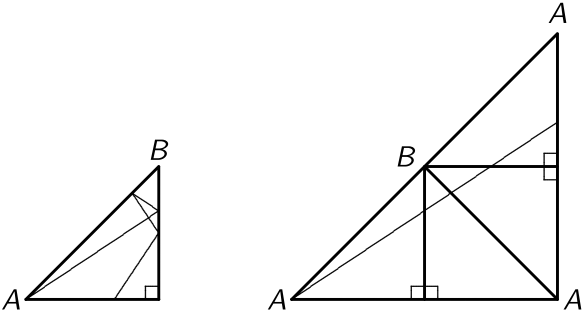

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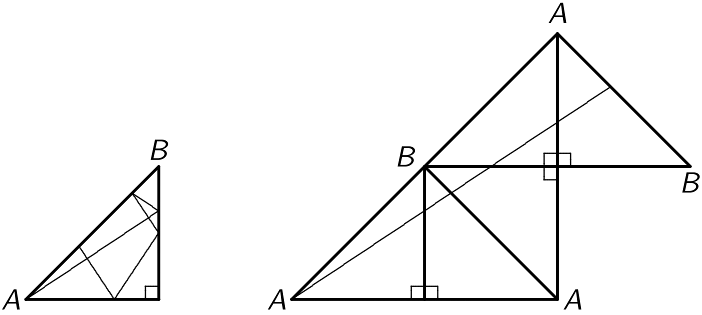

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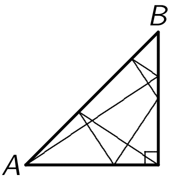

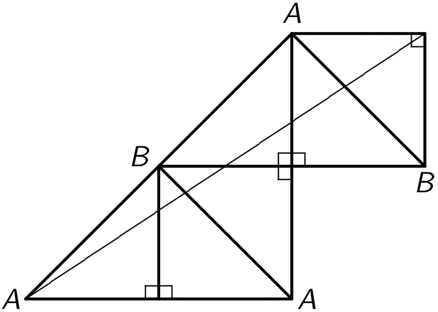

In an isosceles right triangle $\triangle ABC$ (with the right angle at C), no billiard path starting at A can return to A [Tok95]

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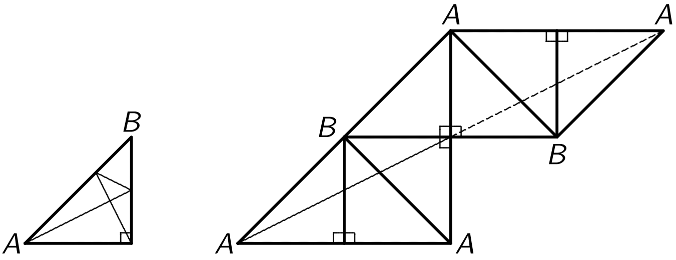
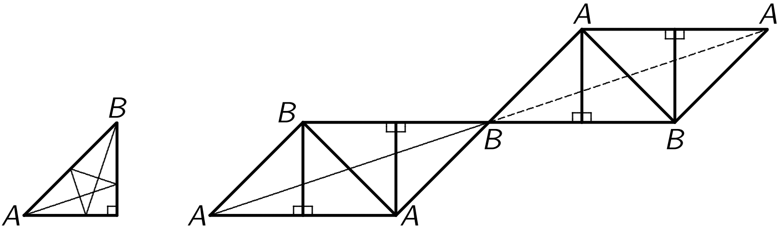


The path starts at A but cannot return to A because it hits C



The path starts at A but cannot return to A because it hits B

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e.g., [Rui91, p. 960]

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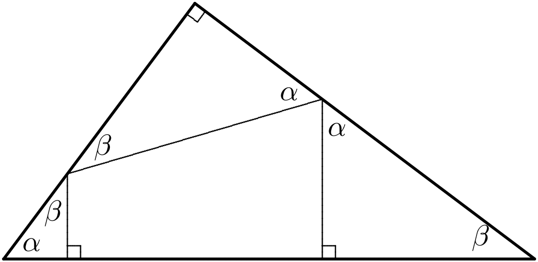

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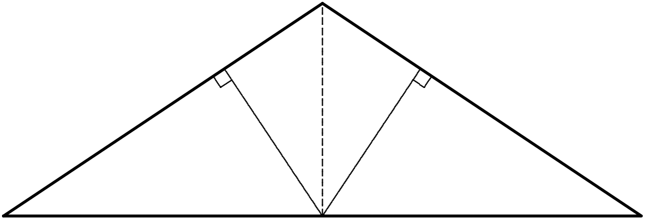

Every acute triangle has a periodic billiard path

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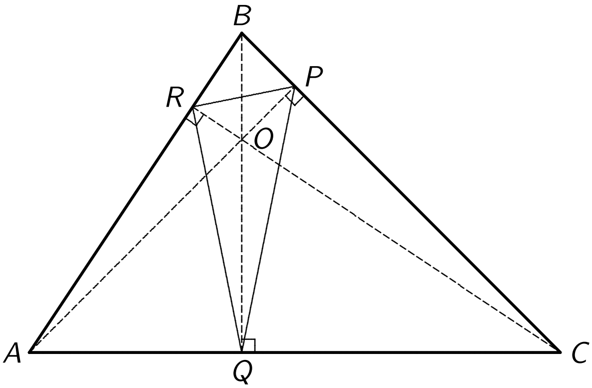
The *orthic* triangle of the given acute triangle is the triangle whose vertices are the bases of the altitudes of the given acute triangle



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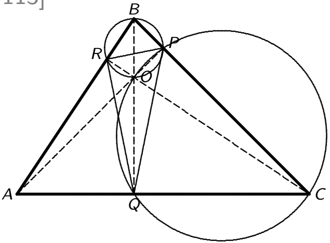
Sketch of a proof [Tab05, p. 113]



BPOR has two right angles; hence it is inscribed into a circle

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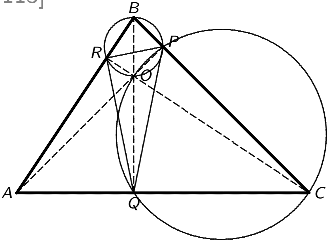

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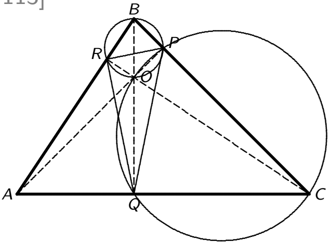

- *BPOR* has two right angles; hence it is inscribed into a circle
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- Likewise, $\angle APQ = \angle ACR$

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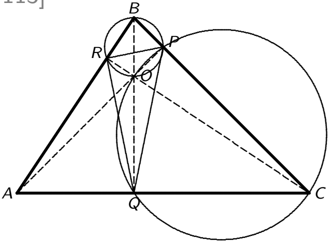

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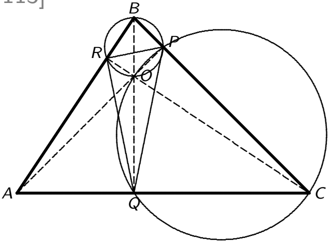

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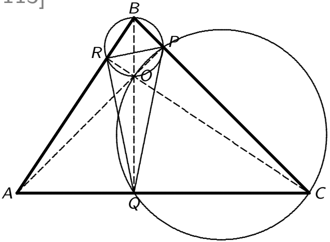

- $BPOR$ has two right angles; hence it is inscribed into a circle
- $\angle APR = \angle ABQ$ because the same arc of this circle supports them
- Likewise, $\angle APQ = \angle ACR$
- $\angle ABQ = \angle ACR$ because both complement $\angle BAC$ to $\pi/2$
- Thus, $\angle APR = \angle APQ$
- Similarly, $\angle BQP = \angle BQR$ and $\angle CRQ = \angle CRP$

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A more detailed sketch of a proof is in [Sch06, p. 3]

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An exercise

Let a square be in a rectangular coordinate system with its sides parallel to the coordinate axes. Let a billiard path in the square start at a vertex. If the path starts at a rational slope, then it is a finite path (that is, it ends at a vertex). Otherwise, it is a semi-infinite path.

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This can easily be proven by unfolding.

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- Do all obtuse triangles have periodic billiard paths?

  Certain obtuse triangles have periodic perpendicular paths [HH00]

  Periodic paths exist for all triangles with angles that are at most 100° [Sch09]

Michael D. Boshernitzan.
Billiards and rational periodic directions in polygons.
The American Mathematical Monthly, 99(6):522–529, 1992.

Andrew M. Baxter and Ronald Umble.
Periodic orbits for billiards on an equilateral triangle.
The American Mathematical Monthly, 115(6):479–491, 2008.

Eugene Gutkin.
Billiards in polygons: Survey of recent results.
Journal of Statistical Physics, 83(1/2):7–26, 1996.

Lorenz Halbeisen and Norbert Hungerbühler.
On periodic billiard trajectories in obtuse triangles.
SIAM Review, 42(4):657–670, 2000.

Th. W. Ruijgrok.
Periodic orbits in triangular billiards.
Acta Physica Polonica B, 22(11–12):955–981, 1991.

Richard Evan Schwartz.
Billiards obtuse and irrational, 2006.

Richard Evan Schwartz.
Obtuse triangular billiards II: One hundred degrees worth of periodic trajectories.
Experimental Mathematics, 18(2):137–171, 2009.

Serge Tabachnikov.
Geometry and Billiards.
American Mathematical Society, 2005.

George W. Tokarsky.
Polygonal rooms not illuminable from every point.
The American Mathematical Monthly, 102(10):867–879, 1995.