Periodic Billiard Paths in Triangles

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Abstract

Let a point move on a frictionless plane bounded by a closed figure. If it hits the boundary, it changes its direction of motion such that the angle of reflection is equal to the angle of incidence. The path that the point follows is called a billiard path. If the point returns to a location with the same direction of motion it had before at that location, then its path is said to be periodic. While it is known that a periodic billiard path exists in every acute triangle and in every right triangle, it is an open problem whether or not every obtuse triangle has a periodic billiard path.
Periodic Billiard Paths in Triangles

Introduction

Definitions

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- If it hits a corner (a vertex), then it stops
- If it hits a side (an edge), then it changes its direction such that the angle of reflection is equal to the angle of incidence
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- If it hits a corner (a *vertex*), then it stops
- If it hits a side (an *edge*), then it changes its direction such that the angle of reflection is equal to the angle of incidence
- The path that the point follows is called a *billiard path*
- An *infinite* path neither starts nor ends at a vertex
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- A periodic path is one whose point returns to a location with the same direction of motion it had before at that location
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- A semi-infinite path either starts but does not end at a vertex, or ends but does not start at a vertex.
- A periodic path is one whose point returns to a location with the same direction of motion it had before at that location.
- A perpendicular path is one whose point hits an edge at a right angle.
Unfolding
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In an isosceles right triangle $\triangle ABC$ (with the right angle at $C$), no billiard path starting at $A$ can return to $A$ [Tok95]
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The path starts at $A$ but cannot return to $A$ because it hits $C$
The path starts at $A$ but cannot return to $A$ because it hits $B$.
Every right triangle has a periodic billiard path
e.g., [Rui91, p. 960]
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Every isosceles triangle has a periodic billiard path
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Every acute triangle has a periodic billiard path
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The *orthic* triangle of the given acute triangle is the triangle whose vertices are the bases of the altitudes of the given acute triangle.
Every acute triangle has a periodic billiard path

Sketch of a proof \cite{Tab05, p. 113}

$BPOR$ has two right angles; hence it is inscribed into a circle
Every acute triangle has a periodic billiard path

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$BPOR$ has two right angles; hence it is inscribed into a circle

$\angle APR = \angle ABQ$ because the same arc of this circle supports them
Every acute triangle has a periodic billiard path

Sketch of a proof [Tab05, p. 113]

\[ \triangle APR \text{ and } \triangle ABQ \text{ have a right angle at } R \text{ and } Q \text{ respectively.} \]

\[ \angle APR = \angle ABQ \text{ because the same arc of the circle supports both angles.} \]

\[ \angle APQ = \angle ACR \text{ because both complement } \angle BAC \text{ to } \pi/2. \]

\[ BPOR \text{ has two right angles; hence it is inscribed into a circle.} \]
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Sketch of a proof [Tab05, p. 113]

\[ BPOR \text{ has two right angles; hence it is inscribed into a circle} \]
\[ \angle APR = \angle ABQ \text{ because the same arc of this circle supports them} \]
Likewise, \[ \angle APQ = \angle ACR \]
\[ \angle ABQ = \angle ACR \text{ because both complement } \angle BAC \text{ to } \pi/2 \]
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\[\triangle BPOR\] has two right angles; hence it is inscribed into a circle

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\[\angle ABQ = \angle ACR\] because both complement \(\angle BAC\) to \(\pi/2\)

Thus, \(\angle APR = \angle APQ\)
Every acute triangle has a periodic billiard path

Sketch of a proof [Tab05, p. 113]

$BPOR$ has two right angles; hence it is inscribed into a circle

$\angle APR = \angle ABQ$ because the same arc of this circle supports them

Likewise, $\angle APQ = \angle ACR$

$\angle ABQ = \angle ACR$ because both complement $\angle BAC$ to $\pi/2$

Thus, $\angle APR = \angle APQ$

Similarly, $\angle BQP = \angle BQR$ and $\angle CRQ = \angle CRP$
Every rational triangle has a periodic billiard path

(A *rational triangle* is one whose angles are rational multiples of $\pi$)
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One such path is a perpendicular path that doesn’t hit a vertex
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Any path on a rational triangle has only finitely many possible directions

A path perpendicular to an edge will hit the same edge perpendicularly again, thus it is periodic
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Any path on a rational triangle has only finitely many possible directions.

A path perpendicular to an edge will hit the same edge perpendicularly again, thus it is periodic.

A more detailed sketch of a proof is in [Sch06, p. 3]

A detailed proof is in [Bos92]
An exercise

Let a square be in a rectangular coordinate system with its sides parallel to the coordinate axes. Let a billiard path in the square start at a vertex. If the path starts at a rational slope, then it is a finite path (that is, it ends at a vertex). Otherwise, it is a semi-infinite path.
An exercise

Let a square be in a rectangular coordinate system with its sides parallel to the coordinate axes. Let a billiard path in the square start at a vertex. If the path starts at a rational slope, then it is a finite path (that is, it ends at a vertex). Otherwise, it is a semi-infinite path.

This can easily be proven by unfolding.
Do all triangles have periodic billiard paths?
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- Do all scalene triangles have periodic billiard paths?
Do all triangles have periodic billiard paths?

- Do all scalene triangles have periodic billiard paths?
- Do all irrational triangles have periodic billiard paths?
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- Do all scalene triangles have periodic billiard paths?
- Do all irrational triangles have periodic billiard paths?
  Are all perpendicular paths either periodic or finite? [Rui91] [Gut96, p. 24]
Do all triangles have periodic billiard paths?

- Do all scalene triangles have periodic billiard paths?
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  Are all perpendicular paths either periodic or finite? [Rui91] [Gut96, p. 24]
- Do all obtuse triangles have periodic billiard paths?
Do all triangles have periodic billiard paths?

- Do all scalene triangles have periodic billiard paths?
- Do all irrational triangles have periodic billiard paths?
  
  Are all perpendicular paths either periodic or finite? [Rui91] [Gut96, p. 24]

- Do all obtuse triangles have periodic billiard paths?
  
  Certain obtuse triangles have periodic perpendicular paths [HH00]
  
  Periodic paths exist for all triangles with angles that are at most $100^\circ$ [Sch09]
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Billiards and rational periodic directions in polygons.

Andrew M. Baxter and Ronald Umble.
Periodic orbits for billiards on an equilateral triangle.

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Billiards in polygons: Survey of recent results.

Lorenz Halbeisen and Norbert Hungerbühler.
On periodic billiard trajectories in obtuse triangles.

Th. W. Ruijgrok.
Periodic orbits in triangular billiards.

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Billiards obtuse and irrational, 2006.

Richard Evan Schwartz.
Obtuse triangular billiards II: One hundred degrees worth of periodic trajectories.

Serge Tabachnikov.
Geometry and Billiards.
George W. Tokarsky.

**Polygonal rooms not illuminable from every point.**