On Derangement and
“Extended” Stirling Number of the First Kind

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Abstract

We developed an extension of Stirling Number of the first kind. It is shown that the number of derangements can be determined using an extension of Stirling number of the first kind.

1 Introduction

Let \( N = \{1, 2, \ldots, n\} \). A permutation \((a_1, a_2, \ldots, a_n)\) is a derangement if \( a_i \neq i \) for \( i = 1, 2, \ldots, n \) where \( a_i \in N \). We can also view a derangement as an automorphism \( f \) on \( N \) such that \( f(i) \neq i \) for each \( i = 1, 2, \ldots, n \). The number of such permutation in \( N \) is denoted by \( D_n \). In this paper, the author will investigate some of the relationships of derangements on \( N \) with the “Extended” Stirling number of the first kind. The Stirling number \( s(n, r) \) of the first kind is the number of ways of distributing \( n \) distinct objects into \( r \) indistinguishable circular tables such that each table has at least one object. The “Extended” Stirling number of the first kind will be defined in the next section. In particular, the author will determine the count of the set of derangements on \( N \) using “Extended” Stirling number \( s^{(2)}(n, r) \) of the first kind, thus treating \( D_n \) as a distribution problem.

2 Stirling Number of the First Kind

Definition 1. Let \( r, n \in \mathbb{Z} \) with \( 0 \leq r \leq n \). The Stirling number \( s(n, r) \) of the first kind is the number of ways to arrange \( n \) distinct objects around \( r \) identical tables such that each table has at least one object.

Definition 2 (Extended Stirling number of the First Kind). Let \( r, n \in \mathbb{Z} \) with \( 0 \leq r \leq n \) and \( 1 \leq k \leq n \). The extended Stirling number \( s^{(k)}(n, r) \) of the first kind is the number of ways of distributing \( n \) distinct objects into \( r \) indistinguishable circular tables such that each table has at least \( k \) objects.

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Remark 1. $s^{(1)}(n, r) = s(n, r)$

Theorem 1. Let $r, n \in \mathbb{Z}$ with $0 \leq r \leq n$. Then

(i) $s(n, 2) = (n - 1)! \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1}\right)$

(ii) $s(n, n - 1) = \binom{n}{2}$

(iii) $s(n, n - 2) = \frac{1}{24} n(n-1)(n-2)(3n-1)$

(iv) $\sum_{r=0}^{n} s(n, r) = n!$

Proof.

(i) By definition, $s(n, 2)$ is the number of ways to arrange $n$ distinct objects into 2 indistinguishable tables. For each $i \in \{1, 2, \ldots, r\}$ we can arrange $i$ distinct objects into two indistinguishable circular tables into $\binom{n}{i} (i-1)! (n-1)!$ ways. The partition $\{i, n-i\}$ and $\{n-i, i\}$ are the same since the tables are indistinguishable, thus $s(n, 2) = \frac{1}{2} \sum_{i=1}^{n-1} \binom{n}{i} (i-1)! (n-1)! = (n-1)! \sum_{i=1}^{n-1} \frac{1}{7}$ as desired.

(ii) To arrange $n$ objects into $n-1$ indistinguishable tables we have only one configuration that is possible: one table will contain exactly two objects and the rest of the tables will have exactly one object. Since the tables are indistinguishable we just need to count the table that will contain two objects. Clearly, there will be $\binom{n}{2}$ possible arrangements. Therefore $s(n, 2) = \binom{n}{2}$.

(iii) Since there are $n$ objects and $n-2$ identical circular tables, then we have the following configurations:

Configuration 1: One table contains 3 objects and the rest will contain only one object.

Configuration 2: Two tables will contain 2 objects and the rest will contain only 1 object.

In configuration 1, we will have $(3-1)! \cdot \binom{n}{3}$ arrangements, since we only need to count the table which contains 3 objects and each configuration can be arranged in $(3-1)!$ ways.

In configuration 2, we will have $\frac{4}{2} \cdot \binom{n}{4}$ arrangements, since we need 4 objects in this configuration and these 4 objects can be arranged in $\frac{4}{2}$ into 2 circular tables. Thus,

$$s(n, n-2) = (3-1)! \binom{n}{3} + \frac{\binom{n}{4}}{2}$$

$$= \frac{n(n-1)(n-2)}{3} + \frac{n(n-1)(n-2)(n-3)}{8}$$

$$= \frac{n(n-1)(n-2)(n-3)(3n-1)}{24}$$

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(iv) We will borrow some results in group theory to prove the last identity. Every permutation element in the symmetric group can be written as product of disjoint cycles. Thus, an arrangement of $n$ distinct objects into $n$ indistinguishable tables is just a permutation element in $S_n$, the symmetric group on $n$ objects. Similarly, every arrangement of $n$ distinct objects into $r$ indistinguishable tables is a permutation element by treating the circular tables as cycles. Hence, it follows that since $|S_n| = n!$, then $\sum_{r=1}^{n} s(n, r) = |S_n| = n!$.

**Theorem 2.** $s(n, r) = s(n - 1, r - 1) + (n - 1)s(n - 1, r)$

**Proof.** We can denote the $n$ distinct objects as $1, 2, \ldots, n$. Consider the object “1”. In any arrangement, we can have 2 cases.

Case 1: “1” is the only object in the circle.

Case 2: “1” is mixed with others in a circle.

In case 1, there are $s(n - 1, r - 1)$ to form such arrangements.

In case 2, the $n - 1$ objects are put in $r$ circles in $s(n - 1, r)$ ways. Then, “1” can be placed to the “immediate right” of the corresponding $n - 1$ objects. By the multiplication principle, there are $(n - 1)s(n - 1, n)$ ways to form such arrangements in case 2. The identity follows from the definition of $s(n, r)$ and by addition principle.

**Theorem 3.** $x(x + 1)(x + 2) \cdots (x + n - 1) = \sum_{r=0}^{n} s(n, r)x^r$

**Proof.** We prove it by induction on $n$. Consider

\[
\prod_{i=0}^{n-1} (x + i) = \sum_{r=0}^{n} s(n, r)x^r
\]

\[
(x + n) \prod_{i=0}^{n-1} (x + i) = (x + n) \sum_{r=0}^{n} s(n, r)x^r
\]

\[
= s(n, 0) + \sum_{r=1}^{n} (s(n, r) + ns(n, r + 1))x^r + s(n, n)x^{n+1}
\]

\[
\prod_{i=0}^{n} (x + i) = s(n, 0) + \sum_{r=1}^{n} s(n + 1, r)x^r + s(n, n)x^{n+1}
\]

\[
= s(n + 1, 0) + \sum_{r=1}^{n} (s(n, r) + ns(n, r - 1))x^r + s(n + 1, n + 1)x^{n+1}
\]

\[
= \sum_{r=0}^{n+1} s(n + 1, r)x^r
\]
3 Derangement

Definition 3. Let \( N_n = \{1, 2, \ldots, n\} \). If \( \sigma = (a_1, a_2, \ldots, a_n) \) is a permutation of \( N_n \), then \( \sigma \) is called a derangement if \( a_i \neq i \) for \( i = 1, 2, \ldots, n \). The number of derangements on \( N_n \) is given by \( D_n \).

Definition 4. Let \( S \) be a set such that \(|S| = n\) where \( n \in \mathbb{N} \) and \( r \leq n \). If \( a_i \in S \) for each \( i = 1, \ldots, r \) then a permutation \( (a_1, a_2, \ldots, a_r) \) \( S \) is called an \( r \)-permutation on \( S \).

Definition 5. Let \( 0 \leq r \leq k \). \( D(n, r, k) \) is the number of \( r \)-permutations on \( N_n \) that have exactly \( k \) fixed points.

Definition 6. Let \( n \in \mathbb{N} \). Define \( S_n \) be the group of permutations on \( n \) objects.

4 Main Results

Lemma 1. Let \( \sigma \) be a permutation of \( N_n \). Then \( \sigma \) is a derangement if and only if \( \sigma \) can be written as product of disjoint cycles such that each cycle is of order at least 2 and the sum of the orders of these cycles is equal to \( n \).

Proof.
\( \Rightarrow \)
Let \( \sigma \in S_n \) be a derangement. We can write \( \sigma = \tau_1 \tau_2 \cdots \tau_k \), for some positive integer \( k \), as product of disjoint cycles. Since the image of each element in \( N_n \) is not equal to itself, then each element belongs to a unique \( \tau_i \) for some \( i \) with \(|\tau_i| \geq 2\). Thus, \( \sum_{i=1}^{k} |\tau_i| = n \) for \( i = 1, 2, \ldots, k \).

\( \Leftarrow \)
Suppose \( \sigma \in S_n \). Let \( x \in N_n \). Thus \( x \) belongs to one of the cycles since \( \sigma \) can be written as product of disjoint cycles with the sum of the orders of the cycles is \( n \). Thus, the image of \( x \) is not itself since the order of the cycle containing \( x \) is at least 2. Since \( x \) was arbitrary, then it follows that \( \sigma \) is a derangement.

Theorem 4. \( D_n \) is equal to the number of ways of distributing \( n \) distinct objects into \( r \) indistinguishable circular tables such that each table contains at least two objects for each \( r = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor \). That is,
\[
D_n = \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor} s^{(2)}(n, r)
\]

Proof. Let \( \sigma \) be a derangement on \( N_n \). Then by Lemma 1, \( \sigma \) can be written as product of disjoint cycles say \( \sigma = \tau_1 \tau_2 \cdots \tau_k \) whose order of each cycle is at least 2 and the sum of the orders of the cycles is \( n \). A cycle can be written as an arrangement of the objects of in a circular table. Then \( \sigma \) can be viewed as an arrangement of \( n \) distinct objects into \( r \) circular tables such that each table has at least two objects for each \( r = 1, 2, \ldots, n \). The tables are indistinguishable since these cycles commute because the cycles are disjoint.
Hence, $D_n$ is equal to the number of ways of distributing $n$ distinct objects into $r$ indistinguishable circular tables such that each table has at least two objects for $r = 1, 2, \cdots, n$. But if $r > \left\lfloor \frac{n}{2} \right\rfloor$ no such arrangement is possible since there will be tables that will contain at most 1 object, so a possible arrangement is valid for $r = 1, 2, \cdots, \left\lfloor \frac{n}{2} \right\rfloor$. Therefore, $D_n = \sum_{r=1}^{\left\lfloor \frac{n}{2} \right\rfloor} s^{(2)}(n, r)$.

**Theorem 5.**

$$D(r, r, k) = \binom{r}{k} \sum_{t=0}^{\left\lfloor \frac{r-k}{2} \right\rfloor} s^{(2)}(r-k, t)$$

**Proof.** By definition $D(r, r, k)$ is the number of permutations on $N_r$ with $k$ fixed points where $k \leq r$. By fixing $k$ such points, there will be $r-k$ points to be deranged. Since there are $r$ objects, then there are $\binom{r}{k}$ ways of fixing $k$ points in $r$ objects. The number of derangements for each $k$ fixed points is equal to $\sum_{t=0}^{\left\lfloor \frac{r-k}{2} \right\rfloor} s^{(2)}(r-k, t)$, by Theorem 4. So, we will have

$$D(r, r, k) = \binom{r}{k} \sum_{t=0}^{\left\lfloor \frac{r-k}{2} \right\rfloor} s^{(2)}(r-k, t)$$

**References**